The Science of Finance

ISDA CDS Standard Model

Proposed Numerical Fix \ Thursday, November 15, 2012

Purpose of the Document

This document is intended to detail a minor deficiency in the currently available ISDA CDS Standard Model code, and propose a solution. It should serve as a discussion point for reviewing the deficiency, solution, and determine a path forward.

Numerical Fix on ISDA Model

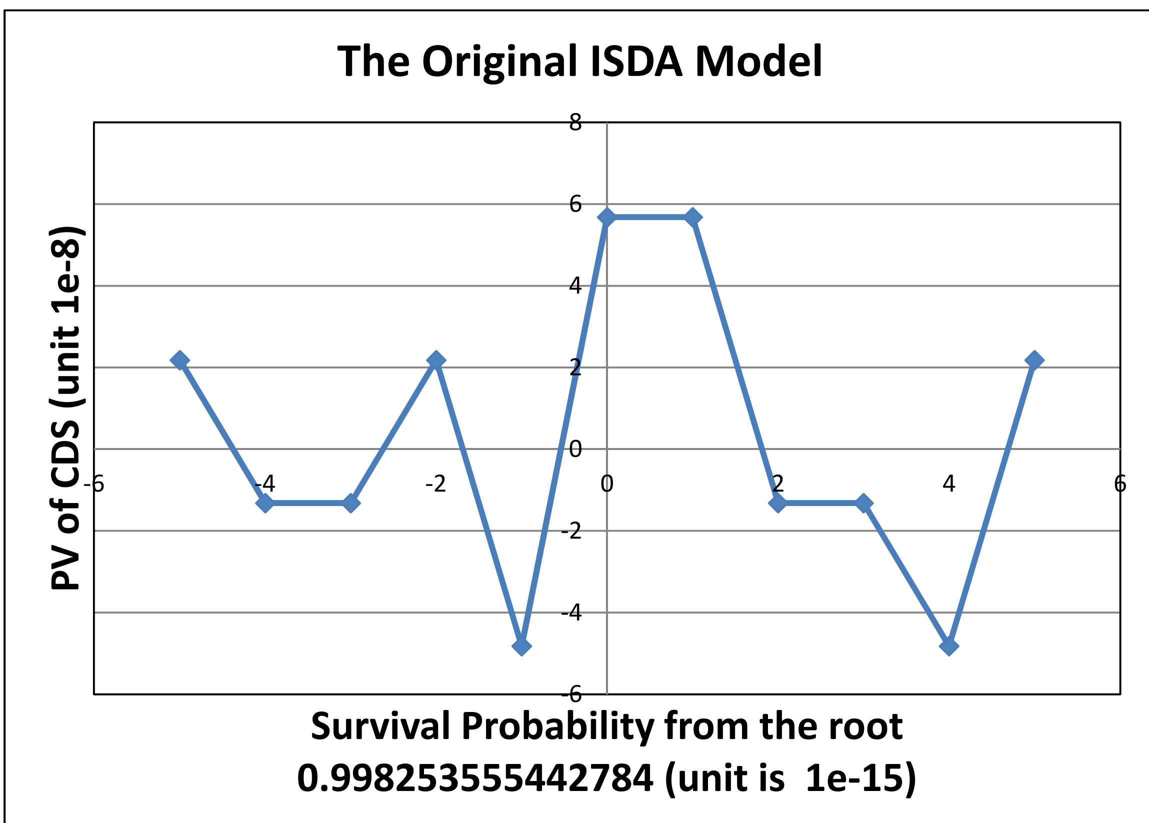
The original ISDA Model calculates the accrual on default for each sub period using the following formula,

$$A = \lambda S_0 D_0 \left[\frac{t_0 + \frac{1}{\lambda+f}}{\lambda+f} - \frac{t_1 + \frac{1}{\lambda+f}}{\lambda+f} \frac{S_1}{S_0} \frac{D_1}{D_0} \right],$$

where t_0 and t_1 are start and end time of the period; S_0, S_1 and D_0, D_1 are the survival probabilities and discount factors until t_0 and t_1; $\lambda = \frac{\ln\left(\frac{S_0}{S_1}\right)}{t_1 - t_0}$ is the forward intensity and $f = \frac{\ln\left(\frac{D_0}{D_1}\right)}{t_1 - t_0}$ is the forward interest rate.

In the case of very small values of $\lambda + f$ numerical noise might arise. This becomes very serious in the case of negative forward interest rates f.

The PV of CDS shall be smooth as a function of default probability. The following diagram is a result of numerical issues arising from the division by the number close to zero in the above formula.



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We can see that the function has big numerical noises and crosses zero line more than once. To remove the noise, we need to transform the equations and remove $\lambda + f$ from denominators. In order to make the implementation more robust, let us define:

$$\Lambda = \lambda(t_1 - t_0) = \lambda t$$
$$F = f(t_1 - t_0) = ft$$

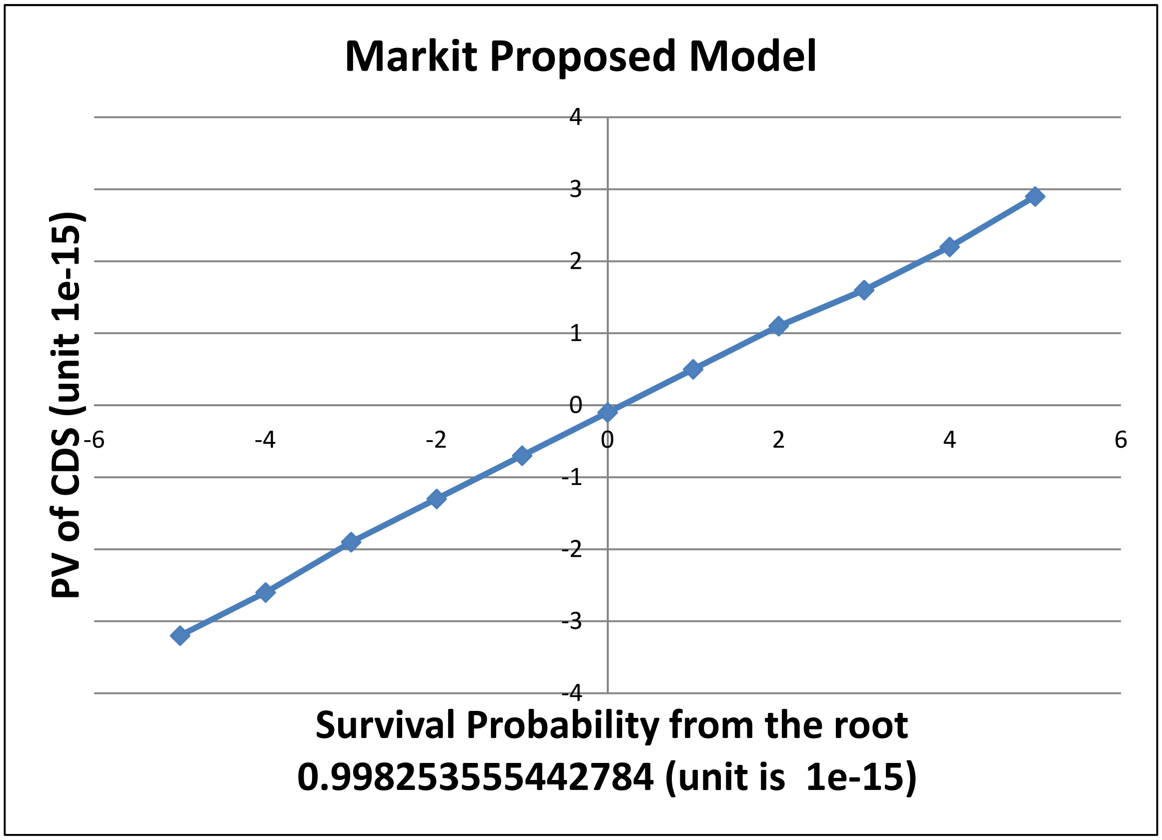
Then the accrual factor A can be expanded in terms of small $\Lambda + F$ as:

$$\begin{aligned}
A = {} & \lambda S_0 D_0 \left[\frac{t_0 + \frac{1}{\lambda+f}}{\lambda + f} - \frac{t_1 + \frac{1}{\lambda+f}}{\lambda + f} \frac{S_1}{S_0} \frac{D_1}{D_0} \right] \\
= {} & \Lambda S_0 D_0 \left[\frac{t_0 + \frac{t}{\Lambda+F}}{\Lambda + F} - \frac{t_1 + \frac{t}{\Lambda+F}}{\Lambda + F} e^{-(\Lambda+F)} \right] \\
= {} & \Lambda S_0 D_0 \Bigg[\frac{1}{\Lambda + F} \Big\{ t_0 - t_1 \Big[1 - (\Lambda + F) + \frac{1}{2}(\Lambda + F)^2 - \frac{1}{6}(\Lambda + F)^3 + \frac{1}{24}(\Lambda + F)^4 - \frac{1}{120}(\Lambda + F)^5 + O((\Lambda + F)^6) \Big] \Big\} \\
& \qquad + \frac{t}{(\Lambda + F)^2} \Big[(\Lambda + F) - \frac{1}{2}(\Lambda + F)^2 + \frac{1}{6}(\Lambda + F)^3 - \frac{1}{24}(\Lambda + F)^4 + \frac{1}{120}(\Lambda + F)^5 \\
& \qquad + O((\Lambda + F)^6) \Big] \Bigg] \\
= {} & \Lambda S_0 D_0 \Bigg[t_1 \Big[1 - \frac{1}{2}(\Lambda + F) + \frac{1}{6}(\Lambda + F)^2 - \frac{1}{24}(\Lambda + F)^3 + \frac{1}{120}(\Lambda + F)^4 + O((\Lambda + F)^5) \Big] \\
& \qquad + t \Big[-\frac{1}{2} + \frac{1}{6}(\Lambda + F) - \frac{1}{24}(\Lambda + F)^2 + \frac{1}{120}(\Lambda + F)^3 - \frac{1}{720}(\Lambda + F)^4 + O((\Lambda + F)^5) \Big] \Bigg] \\
= {} & \Lambda S_0 D_0 \left[\frac{t_0 + t_1}{2} - \frac{t_0 + 2t_1}{6}(\Lambda + F) + \frac{t_0 + 3t_1}{24}(\Lambda + F)^2 - \frac{t_0 + 4t_1}{120}(\Lambda + F)^3 + \frac{t_0 + 5t_1}{720}(\Lambda + F)^4 + O((\Lambda + F)^5) \right]
\end{aligned}$$

Similar expansion can be applied to contingent leg calculation. If R is a recovery rate, the PV of the contingent leg in one period is expressed as,

$$\begin{aligned}
& (1 - R) \frac{\lambda}{\lambda + f} S_0 D_0 \left(1 - \frac{S_1 D_1}{S_0 D_0} \right) \\
& = (1 - R) \frac{\Lambda}{\Lambda + F} S_0 D_0 [1 - \exp(-(\Lambda + F))] \\
& = (1 - R) \frac{\Lambda}{\Lambda + F} S_0 D_0 \left[(\Lambda + F) - \frac{1}{2}(\Lambda + F)^2 + \frac{1}{6}(\Lambda + F)^3 - \frac{1}{24}(\Lambda + F)^4 + \frac{1}{120}(\Lambda + F)^5 + \cdots \right] \\
& = (1 - R) \Lambda S_0 D_0 \left[1 - \frac{1}{2}(\Lambda + F) + \frac{1}{6}(\Lambda + F)^2 - \frac{1}{24}(\Lambda + F)^3 + \frac{1}{120}(\Lambda + F)^4 + \cdots \right]
\end{aligned}$$

With these fixes implemented, the PV of the same CDS contract looks like:



A Test Case

t0 = 0.0972602739726;

t1 = 0.25616438356164;

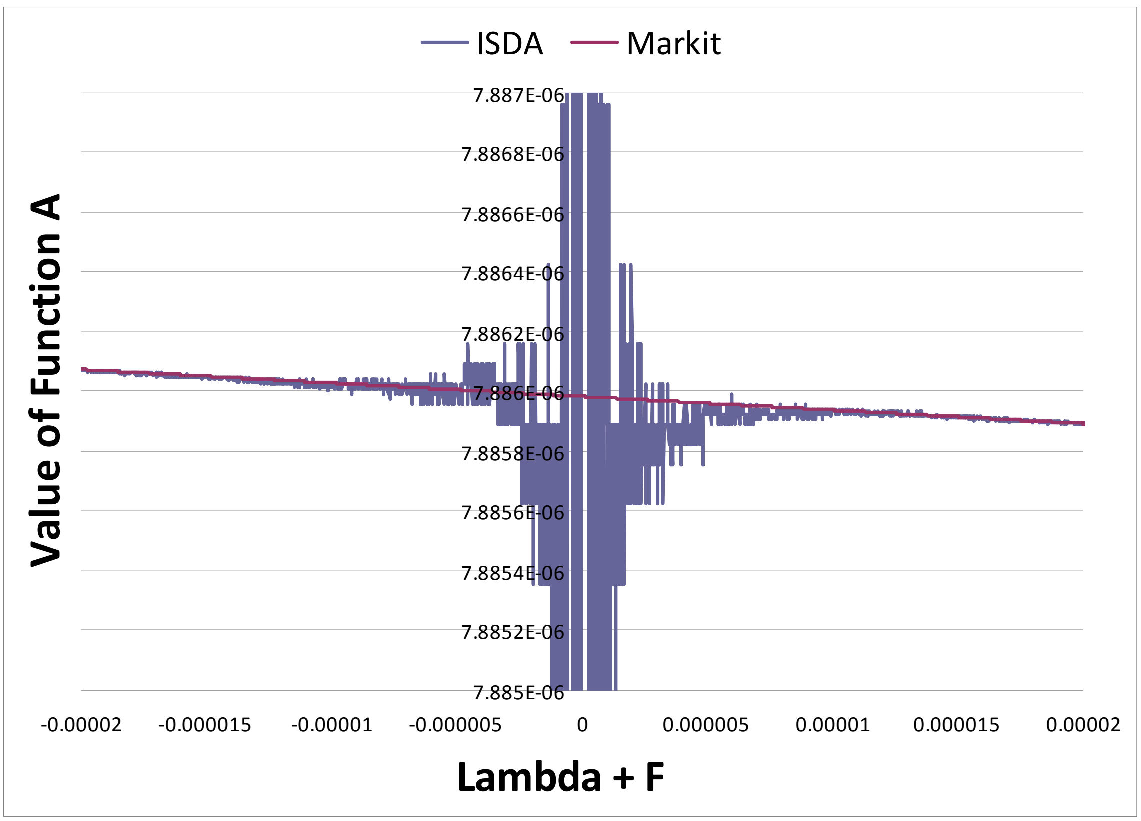
s0 = 0.99853738025987;

df0 = 0.98929526837792;

lambda = 0.00028429134225;

fwdRate = -0.00028433979274;

We plot the value of A when $\Lambda + F$ is close to zero.



Here the we use an expansion up to $4th$ order of $\Lambda + F$.

We observe that the original method contains much more noise than the expansion method.

We increase the range of our test to cover the full range in which the expansion is in effect. In our implementation, the criteria we set to use the expansion is $|\Lambda + F| < 1 \times 10^{-4}$. The diagram below shows that the original method and the expansion method are consistent and continuous around the boundary.